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Oscillation and Asymptotic Behavior of Solutions of Differential Equations with Retarded Argument

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1. INTRODUCTION

In this paper we are concerned with the oscillation and asymptotic behavior of solutions of the nonlinear differential equation with retarded argument

$$y^{(n)}(t) + f(t, y(t), y(g(t))) = 0, \tag{1.1}$$

where

$$f \in C[[0, \infty) xRxR, R]; \quad g \in C[[0, \infty), R] \tag{1.2}$$

and

$$g(t) \leqslant t, \quad t \geqslant 0; \qquad \lim_{t\to\infty} g(t) = \infty. \tag{1.3}$$

We shall assume that under the initial conditions

$$y(t) = \phi(t), \quad t \leqslant t_0 \qquad \text{and} \qquad y^{(k)}(t_0) = y_k, \quad k = 1, 2,..., n-1, \tag{1.4}$$

Eq. (1.1) has a solution which exists for all $t \geqslant t_0$. In general, by a solution of the equation (1.1), we shall understand a function $y \in C^n[[t_0, \infty), R]$ which satisfies (1.1) for all $t \geqslant t_0$ for some $t_0 \geqslant 0$. We call a solution $y(t)$ of Eq. (1.1) *oscillatory* if it has no last zero, i.e., if $y(t_0) = 0$ for some $t_0 \geqslant 0$ then there exists a $t_1 > t_0$ for which $y(t_1) = 0$. We call a solution $y(t)$ of Eq. (1.1) *nonoscillatory* if it is eventually of constant sign. Equation (1.1) is called oscillatory if every solution is oscillatory.

There are very few references in English in oscillation results for functional differential equations although, these results are of great importance both in theory and applications. The only papers known to this author are [6, 9, 14] dealing with a nonlinear first-order equation, and [4, 12, 13, 15] dealing with second-order delay equations. We should also mention the books of S. B. Norkin [10] and R. Bellman and K. Cooke [2] and the lecture notes of J. Hale [5].

In Section 2 we prove a theorem on the asymptotic behavior of the solutions of Eq. (1.1) which also proves that under certain conditions the Eq. (1.1) is nonoscillatory.

In Section 3 we prove sufficient and necessary conditions for oscillation.

In Section 4 we prove oscillation theorems for bounded solutions of a variant of Eq. (1.1).

2. Here we shall study the asymptotic behavior for $t \to \infty$ of solutions of the nonlinear differential Eq. (1.1). Our basic result is that under appropriate conditions on the function f the Eq. (1.1) has solutions which approach those of $y^{(n)}(t) = 0$ as $t \to \infty$. The special case $n = 2$ and $g(t) = t$ was considered by D. Cohen [3]. Cohen's theorem is a generalization of a fundamental result in Bellman [1, pp. 114–115] which states that if $y(t)$ is a solution of the equation $y'' + p(t)\, y = 0$ and if

$$\int^{\infty} tp(t)\, dt < \infty, \tag{2.1}$$

then $y(t)$ is asymptotic to $a_0 + a_1 t$ as $t \to \infty$. We prove the following extension of this result.

THEOREM 2.1. *Let the functions f and g satisfy the conditions* (1.2) *and* (1.3), *and in addition, suppose that*

$$|f(t, y, z)| \leqslant p(t)\,|y| + q(t)\,|z|, \qquad t \geqslant 0, \quad y, \quad z \in R, \tag{2.2}$$

where $p(t)$ and $q(t)$ are nonnegative continuous functions such that

$$\int^{\infty} \{s^{n-1}p(s) + [g(s)]^{n-1}\, q(s)\}\, ds < \infty. \tag{2.3}$$

Then Eq. (1.1) *has solutions which are asymptotic to $at^{n-1} (a \neq 0)$ as $t \to \infty$.*

Proof. Our proof is essentially an extension of Bellman's proof to higher-order equations. Choose $t_0 \geqslant 1$ so large that $g(t) > 0$, $t \geqslant t_0$. Integrating (1.1) n-times from t_0 to t we obtain

$$y(t) - \sum_{i=0}^{n-1} \frac{y^{(i)}(t_0)}{i!}(t - t_0)^i + \frac{1}{(n-1)!}\int_{t_0}^{t} (t - s)^{n-1} f(s, y(s), y(g(s)))\, ds = 0.$$

From this and in view of (2.2) we get for $t \geqslant t_0$ the estimate

$$|y(t)| \leqslant Ct^{n-1} + t^{n-1}\int_{t_0}^{t} \{p(s)\,|y(s)| + q(s)\,|y(g(s))|\}\, ds, \tag{2.4}$$

where C is a positive constant.

Define the function

$$F(t) = C + \int_{t_0}^{t} \{p(s) \, |y(s)| + q(s) \, |y(g(s))|\} \, ds. \tag{2.5}$$

Then $F(t)$ is nondecreasing in t:

$$|y(t)| \leqslant t^{n-1}F(t), \qquad t \geqslant t_0 \tag{2.6}$$

and

$$F'(t) = p(t) \, |y(t)| + q(t) \, |y(g(t))|. \tag{2.7}$$

Choose $t_1 \geqslant t_0$ so large that $g(t) \geqslant t_0$ for $t \geqslant t_1$. Then from (2.6) and the monotone character of F we obtain

$$|y(g(t))| \leqslant [g(t)]^{n-1}F(t), \qquad t \geqslant t_1 . \tag{2.8}$$

From (2.7) and in view of (2.6) and (2.8) we obtain

$$F'(t) \leqslant \{t^{n-1}p(t) + [g(t)]^{n-1}q(t)\} F(t), \qquad t \geqslant t_1 .$$

Hence

$$\begin{aligned} F(t) &\leqslant F(t_1) \exp \int_{t_1}^{t} \{s^{n-1}p(s) + [g(s)]^{n-1} q(s)\} \, ds \\ &\leqslant F(t_1) \exp \int_{t_1}^{\infty} \{s^{n-1}p(s) + [g(s)]^{n-1} q(s)\} \, ds = C_1 , \end{aligned}$$

where in view of the hypothesis (2.3) C_1 is a finite positive constant. The inequalities (2.6) and (2.8) now become

$$|y(t)| \leqslant C_1 t^{n-1}, \qquad t \geqslant t_1 \tag{2.6$'$}$$

and

$$|y(g(t))| \leqslant C_1[g(t)]^{n-1}, \qquad t \geqslant t_1 . \tag{2.8$'$}$$

Integrating (1.1) from t_1 to t we obtain

$$y^{(n-1)}(t) = y^{(n-1)}(t_1) - \int_{t_1}^{t} f(s, y(s), y(g(s))) \, ds. \tag{2.9}$$

In view of (2.2), (2.3), (2.6)$'$ and (2.8)$'$, the integral in (2.9) converges as $t \to \infty$ and therefore the $\lim_{t\to\infty} y^{(n-1)}(t)$ exists and is a finite number. To ensure that this limit is not zero choose t_1 so large that

$$1 - C_1 \int_{t_1}^{\infty} \{s^{n-1}p(s) + [g(s)]^{n-1} q(s)\} \, ds > 0$$

and impose the condition $y^{(n-1)}(t_1) = 1$ on the solution $y(t)$ of Eq. (1.1). This solution $y(t)$ has the desired asymptotic property. The proof is therefore complete.

A similar theorem dealing with linear equations only was proved in [16].

COROLLARY 2.1. *Under the hypotheses of Theorem 2.1 the Eq. (1.1) has nonoscillatory solutions.*

3. A well-known sufficient condition for oscillation of the second-order linear ordinary differential equation

$$y'' + q(t)\,y = 0, \qquad t \geqslant 0$$

is

$$\int_0^\infty q(t)\,dt = \infty.$$

Recently, Waltman [13] extended this result to retarded differential equations of the form

$$y''(t) + q(t)\,f(y(t), y(g(t))) = 0. \tag{3.1}$$

This result of Waltman's extends also an oscillation theorem of Norkin [10, pp. 149–150].

Here we shall obtain an extension of Waltman's theorem for equations of the form (1.1). The following lemma of Kiguradze [8] will be needed.

LEMMA 3.1. *If $y(t)$ is a function such that it and all its derivatives up to order $(n - 1)$ inclusive, are absolutely continuous and of constant sign in the interval (t_0, ∞), and $y^{(n)}(t)\,y(t) \leqslant 0$, then there is an integer l, $0 \leqslant l \leqslant n - 1$, which is even if n is odd and odd if n is even, so that for $t \geqslant t_0$ we have*

$$y^{(k)}(t)\,y(t) \geqslant 0, \qquad k = 0, 1,..., l,$$

$$(-1)^{n+k-1}y^{(k)}(t)\,y(t) \geqslant 0, \qquad k = l + 1,..., \eta,$$

and

$$|\,y(t)| \geqslant \frac{(t - t_0)^{n-1}}{(n-1)\cdots(n-l)}\,|\,y^{(n-1)}(2^{n-l-1}t)|. \tag{3.2}$$

THEOREM 3.1. *Let the functions f and g satisfy the conditions* (1.2) *and* (1.3), *and, in addition, suppose that*

(i) *$f(t, y, z)$ increases in both y and z;*

(ii) *if y and z have the same sign, then $f(t, y, z)$ has that sign for all sufficiently large t;*

(iii) *for any constant* $C \neq 0$

$$\int^{\infty} f(s, Cs^{n-2}, C[g(s)]^{n-2})\, ds = \pm\infty. \tag{3.3}$$

Then every solution of (1.1) *is either oscillatory or* $y(t)\, y^{(n-2)}(t) < 0$ *for sufficiently large* t. *Moreover, every nonoscillatory solution* $y(t)$ *of* (1.1) *satisfies the order relation*

$$y(t) = 0(t^{n-1}), \qquad as \quad t \to \infty.$$

Proof. Let $y(t)$ be a solution of (1.1) existing on $[t_0, \infty)$. If $y(t)$ is nonoscillatory, we may assume that $y(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. The case $y(t) < 0$ is treated similarly. Since $\lim_{t\to\infty} g(t) = \infty$, there exists a $t_2 \geqslant t_1$ such that $y(g(t)) > 0$ for $t \geqslant t_2$. In view of (ii),

$$y^{(n)}(t) = -f(t, y(t), y(g(t))) < 0, \qquad t \geqslant t_2 .$$

Therefore, $y^{(n-1)}(t)$ is strictly decreasing for $t \geqslant t_2$ and must be positive for large t, say $t \geqslant t_2$ (otherwise two consecutive derivatives of $y(t)$ are negative and $y(t)$ should tend to $-\infty$). So $y^{(n-2)}(t)$ is strictly increasing for $t \geqslant t_3$ and, consequently, is of constant sign for large t, say $t \geqslant t_4 \geqslant t_3$. Now we have to examine two cases.

Case I. $y^{(n-2)}(t) > 0$, $t \geqslant t_4$. Then by Taylor's theorem

$$y(t) = y(t_4) + y'(t_4)(t - t_4) + \cdots + \frac{y^{(n-2)}(\xi)}{(n-2)!}(t - t_4)^{n-2}, \qquad t_4 < \xi < t. \tag{3.4}$$

Since $y^{(n-2)}(t)$ is strictly increasing and positive for $t \geqslant t_4$ we conclude from (3.4) that there is a positive constant C and a sufficiently large $t_5 \geqslant t_4$ such that

$$y(t) \geqslant Ct^{n-2}, \qquad t \geqslant t_5 . \tag{3.5}$$

Again, since $\lim_{t\to\infty} g(t) = \infty$, there is a $t_6 \geqslant t_5$ such that $g(t) \geqslant t_5$ for $t \geqslant t_6$. From (3.5) we then get

$$y(g(t)) \geqslant C[g(t)]^{n-2}, \qquad t \geqslant t_6 . \tag{3.6}$$

Integrating (1.1) once from t_6 to t and using (3.5), (3.6) and (i) we obtain

$$\begin{aligned} y^{(n-1)}(t_6) &= y^{(n-1)}(t) + \int_{t_6}^{t} f(s, y(s), y(g(s)))\, ds \\ &\geqslant \int_{t_6}^{t} f(s, Cs^{n-2}, C[g(s)]^{n-2})\, ds, \qquad t \geqslant t_6 \end{aligned}$$

which contradicts the hypothesis (3.3).

Case II. $y^{(n-2)}(t) < 0$, $t \geqslant t_4$, (of course, this will not happen if $n = 2$, since by assumption $y(t) > 0$). Since $y^{(n-1)}(t)$ is positive and decreasing, the $\lim_{t\to\infty} y^{(n-1)}(t) = L$ exists and it is either positive or zero. We shall prove that $L = 0$. Since $y^{(n)}(t)\, y(t) < 0$, it follows from Kiguradze's formula (3.2) and the decreasing character of $y^{(n-1)}(t)$ that there exist a $t_5' \geqslant t_1$ and a positive constant C such that

$$y(t) \geqslant CLt^{n-1}, \qquad t \geqslant t_5'. \tag{3.5$'$}$$

Also, as in (3.6) there exists a $t_6' \geqslant t_5'$ such that

$$y(g(t)) \geqslant CL[g(t)]^{n-1}. \tag{3.6$'$}$$

Integrating (1.1) once from t_6' to t and using (3.5)′, (3.6)′ and (i), we obtain

$$\begin{aligned} y^{(n-1)}(t_6') &= y^{(n-1)}(t) + \int_{t_6'}^{t} f(s, y(s), y(g(s)))\, ds \\ &\geqslant \int_{t_6'}^{t} f(s, CLs^{n-1}, CL[g(s))]^{n-1})\, ds \\ &\geqslant \int_{t_6'}^{t} f(s, CLs^{n-2}, CL[g(s)]^{n-2})\, ds. \end{aligned} \tag{3.7}$$

From (3.7) and in view of the hypothesis (3.3) we conclude that $L = 0$ and the proof of the theorem is complete.

Remark. If $n = 2$, then the case II of the proof of Theorem 3.1 never occurs and, therefore, for $n = 2$ all solutions of the Eq. (1.1) are oscillatory. This is Waltman's Theorem [13].

The following theorem gives a necessary condition for Eq. (1.1) to be oscillatory.

THEOREM 3.2. *Let the functions f and g satisfy the conditions* (1.2) *and* (1.3); *in addition, suppose that*

(i) *all solutions of Eq.* (1.1) *are oscillatory.*

(ii) $f(t, 0, 0) \geqslant 0$ *and* $f(t, y, z)$ *is nondecreasing in* y *and* z. *Then, for every positive constant* C

$$\int^{\infty} (s, Cs^{n-1}, C[g(s)]^{n-1})\, ds = \infty \tag{3.8}$$

Proof. If the theorem were false, there should exist a constant $C > 0$ such that

$$\int^{\infty} f(s, Cs^{n-1}, C[g(s)]^{n-1})\, ds < \infty. \tag{3.9}$$

Choose t_0, sufficiently large, so that

$$\int_{t_0}^{\infty} f(s, Cs^{n-1}, C[g(s)]^{n-1})\, ds < \frac{C}{2}. \tag{3.10}$$

Construct a solution $y(t)$ of Eq. (1.1) satisfying the initial conditions

$$\begin{aligned} y(t) &\equiv 0, \qquad t \leqslant t_0, \\ y^{(k)}(t_0) &= 0, \qquad k = 1, 2,..., n-2, \end{aligned} \tag{3.11}$$

and

$$y^{(n-1)}(t_0) = C.$$

We claim that this solution of Eq. (1.1) is nonoscillatory contradicting the hypothesis (i). (Recall that it was assumed that every solution of (1.1) satisfying (1.4) exists in the future). Otherwise, let t_1 be the first zero of $y(t)$ in (t_0, ∞). Then, $y(t) \geqslant 0$ and $y(g(t)) \geqslant 0$ for $t \leqslant t_1$. In view of (ii), we then have

$$f(t, y(t), y(g(t))) \geqslant 0, \qquad t \leqslant t_1.$$

Hence,

$$y^{(n)}(t) \leqslant 0, \qquad t \leqslant t_1. \tag{3.12}$$

Integrating (3.12) n-times from t_0 to t, $t_0 \leqslant t \leqslant t_1$, we obtain

$$\begin{aligned} y(t) &\leqslant \frac{C}{(n-1)!}(t-t_0)^{n-1}, \qquad t_0 \leqslant t \leqslant t_1 \qquad \text{and} \\ &= 0 \qquad \text{for} \quad t \leqslant t_0. \end{aligned}$$

Therefore, for $t \leqslant t_1$,

$$y(t) \leqslant Ct^{n-1} \qquad \text{and} \qquad y(g(t)) \leqslant C[g(t)]^{n-1}. \tag{3.13}$$

Integrating (1.1) once from t_0 to t with $t_0 \leqslant t \leqslant t_1$, and using (3.11), (ii), (3.13) and (3.10), we obtain

$$\begin{aligned} y^{(n-1)}(t) &= C - \int_{t_0}^{t} f(s, y(s), y(g(s)))\, ds \\ &\geqslant C - \int_{t_0}^{t} f(s, Cs^{n-1}, C[g(s)]^{n-1})\, ds \\ &\geqslant C - \int_{t_0}^{\infty} f(s, Cs^{n-1}, C[g(s)]^{n-1})\, ds \\ &> C - \frac{C}{2} = \frac{C}{2} > 0. \end{aligned} \tag{3.14}$$

Since $y^{(n-1)}(t) > 0$, $t_0 \leqslant t \leqslant t_1$, the function $y(t)$ which has a zero at t_0 cannot have other zeros for $t > t_0$ (otherwise by Rolle's theorem and (3.11) $y^{(n-1)}(t)$ should have a zero in $[t_0, t_1]$). The proof is therefore complete.

4. Finally, we shall study the oscillatory properties of bounded solutions of the differential equation

$$y^{(n)}(t) + p(t) f(y(t), y(g(t))) = 0, \tag{4.1}$$

where $p(t)$ is a positive continuous function on $[0, \infty)$, $f \in C[RxR, R]$ and $g(t)$ satisfies the conditions (1.2) and (1.3). In the case $n = 2$, Waltman [13] proved that if

$$\int^{\infty} p(t)\, dt = \infty, \tag{4.2}$$

and if the function $f(y, z)$ satisfies the conditions (i) and (ii) of Theorem 3.1., then all the solutions (not necessarily the bounded ones) of Eq. (4.1) are oscillatory. He also gave an example to show that the integral condition (4.2) is not superfluous. The aim here is to restrict our considerations to bounded solutions of Eq. (4.1) so that we will be able to improve the integral condition (4.2) to

$$\int^{\infty} tp(t)\, dt = \infty \tag{4.3}$$

in the case $n = 2$ and to

$$\int^{\infty} t^{n-1} p(t)\, dt = \infty \tag{4.4}$$

in general. The analogous problem for ordinary dIfferential Eqs. was recently treated by A. G. Kartsatos [7] and H. Onose [11]. Our arguments are similar to those in [7].

THEOREM 4.1. *In addition to the hypothesis* (4.4), *assume that if y and z have the same sign, then $f(y, z)$ has that sign. Then,* (a) *for n even, all bounded solutions of the Eq.* (4.1) *are oscillatory.* (b) *for n odd, all bounded solutions of the Eq.* (4.1) *are either oscillatory or tend monotonically to zero.*

Proof. Let $y(t)$ be a bounded solution of the Eq. (4.1). If $y(t)$ is non-oscillatory, then it is of fixed sign in some interval $[t_0, \infty)$. Assume that $y(t) > 0$ for $t \geqslant t_0$. The case $y(t) < 0$ is treated similarly. Since $g(t) \to \infty$ as $t \to \infty$ there exists a $t_1 \geqslant t_0$ such that $y(g(t)) > 0$ for $t \geqslant t_1$. Therefore, $y^{(n)}(t) < 0$ for $t \geqslant t_1$. Since $y(t)$ is a bounded positive function,

$y^{(k)}(t)\, y^{(k+1)}(t) < 0$ for each $k = 1, 2, \ldots, n-1$ and all sufficiently large t, say $t \geqslant t_2 \geqslant t_1$. It follows that

$$(-1)^{k+1} y^{(n-k)}(t) > 0, \qquad k = 0, 1, \ldots, n-1, \qquad t \geqslant t_2. \tag{4.5}$$

Since $y'(t)$ is of fixed sign in $[t_2, \infty)$ it follows that the $\lim_{t\to\infty} y(t) = y(\infty)$ exists and it is bounded (since $y(t)$ is bounded). If n is even, $y(\infty) > 0$ because in this case $y'(t) > 0$. If n is odd, either $y(\infty)$ is zero which proves (b), or $y(\infty) > 0$. So we assume that $y(\infty) > 0$ for n even or odd and we shall prove a contradiction. In view of the continuity of f and the fact that $y(\infty) > 0$ and finite and $\lim_{t\to\infty} y(g(t)) = y(\infty) > 0$, it follows that the limit $L = \lim_{t\to\infty} f(y(t), y(g(t)))$ exists and is a finite positive number. Therefore for t sufficiently large, say, $t \geqslant t_3$ $(L/2) \leqslant f(y(t), y(g(t)))$ and from Eq. (4.1) we obtain

$$y^{(n)}(t) + \frac{L}{2} p(t) \leqslant 0. \tag{4.6}$$

Multiplying both sides of (4.6) by t^{n-1} and integrating from t_3 to t, we obtain

$$\int_{t_3}^{t} s^{n-1} y^{(n)}(s)\, ds + \frac{L}{2} \int_{t_3}^{t} s^{n-1} p(s)\, ds \leqslant 0. \tag{4.7}$$

Successive integration by parts of the first integral in (4.7) gives

$$\int_{t_3}^{t} s^{n-1} y^{(n)}(s)\, ds = P(t) - P(t_3) + (-1)^{n+1} n! [y(t) - y(t_3)], \tag{4.8}$$

where

$$P(t) = \sum_{k=0}^{n-1} (-1)^{k+1} (n-1)(n-2) \cdots (n-k+1)\, t^{n-k} y^{(n-k)}(t)$$

which in view of (4.5) is positive. Since $y(t)$ is bounded and because of the hypothesis (4.4), the inequality (4.7) is impossible. The proof of the theorem is therefore complete.

Remark. Recently, Gollwitzer [4] extended well-known oscillation theorems for the ordinary differential equation

$$y''(t) + g(t)\, y(t)^{\gamma} = 0$$

to the delay equation

$$y''(t) + g(t)\, y_{\tau}(t)^{\gamma} = 0, \tag{4.9}$$

where $g(t) \geqslant 0$, γ is the ratio of odd integers with $0 < \gamma < 1$ or $1 < \gamma$ and

$y_\tau(t)^\gamma \equiv [y(t - \tau(t))]^\gamma$ with $0 \leqslant \tau(t) \leqslant M$. The case $\gamma = 1$ was not examined merely because in this case Theorem 1 in [4] is not valid as the example in [13] shows. However, Theorem 1 in [4] is still true even for $\gamma = 1$ for the bounded solutions of Eq. (4.9) as we proved in Theorem 4.1. In this case the converse is also true and the proof is identical with that given in [4].

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References

1. R. Bellman, "Stability Theory of Differential Equations," McGraw-Hill, New York, 1953.
2. R. Bellman and K. Cooke, "Differential-Difference Equations," Academic Press, New York, 1963.
3. D. S. Cohen, The asymptotic behavior of a class of nonlinear differential equations, *Proc. Amer. Math. Soc.* **18** (1967), 601–603.
4. H. E. Gollwitzer, On nonlinear oscillations for a second-order delay equations, *J. Math. Anal. Appl.* **26** (1969), 385–389.
5. J. K. Hale, "Functional Differential Equations," Lectures at the University of California, Los Angeles, 1968–1969.
6. S. Kakutani and L. Markus, On the nonlinear difference-differential equation $y'(t) = [A + By(t - \gamma)]\, y(t)$, *Contr. Theory Nonlinear Oscillations (Princeton)* **4** (1958), 411–418.
7. A. G. Kartsatos, On oscillation of solutions of even-order nonlinear differential equations, *J. Differential Equations* **6** (1969), 232–237.
8. I. T. Kiguradge, The problem of oscillation of solutions of nonlinear differential equations (Russian), *Differencial'nye Uravnenija* (1) **8** (1965), 995–1006.
9. J. C. Lillo, Oscillatory solutions of the equation $y'(x) = m(x)\, y(x - n(x))$, *J. Differential Equations* **6** (1969), 1–35.
10. S. B. Norkin, Differential equations of the second order with retarded arguments (Russian), Izdat. "Nauka", Moscow, 1965.
11. H. Onose, Oscillatory property of ordinary differential equations of arbitrary order, *J. Differential Equations* **7** (1970), 454–458.
12. K. D. Shere, Nonoscillation of second-order, linear differential equations with retarded argument, to appear.
13. P. Waltman, A note on an oscillation criterion for an equation with a functional argument, *Canad. Math. Bull.* **11** (1968), 593–595.
14. E. M. Wright, A non-linear difference-differential equation, *J. Reine Angew. Math.* **194** (1955), 66–87.
15. W. J. Coles, L. J. Grimm, and K. Schmitt, Oscillation theorems for functional differential equations, to appear.
16. V. I. Logunov, On asymptotic behavior of solutions of the *n*th order linear differential equation with retarded argument (Russian), *Differencial'nye Uravnenija* **1** (1965), 467–468.